LETTER TO THE EDITOR

Quantum integrability and Bethe ansatz solution for interacting matter–radiation systems

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Abstract
A unified integrable system, generating a new series of interacting matter–radiation models with interatomic coupling and different atomic frequencies, is constructed and exactly solved through an algebraic Bethe ansatz. Novel features in Rabi oscillation and vacuum Rabi splitting are shown on the example of an integrable two-atom Buck–Sukumar model with resolution of some important controversies in the Bethe ansatz solution including its possible degeneracy for such models.

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The basic physics underlying a variety of important phenomena in interacting matter–radiation (MR) systems, like those in quantum optics induced by resonance interaction between an atom and a quantized laser field, in cavity QED [1, 2], in a trapped ion interacting with its centre of mass motion irradiated by a laser beam [3, 4] etc, seems to be nicely captured by simple models such as Jaynes–Cummings (JC) [5], Buck–Sukumar (BS) [6] and some of their extensions [7]. Many theoretical predictions based on these models, such as vacuum Rabi splitting (VRS) [2, 8], Rabi oscillation and its quantum collapse and revival [1] etc have been verified in maser and laser experiments. However, for describing physical situations more accurately one has to look for further generalizations of the basic models, such as q-deformed BS and JC models [15, 16], trapped ion (TI) with nonlinear coupling [4, 9], multi-atom models [2, 9, 10] etc. Nevertheless, while the exact solutions for the JC and the BS models together with their simple multi-atom extensions are known [11–13], the same is no longer true for most of the above generalizations. Moreover, while in known multi-atomic MR models the atoms interact only via the oscillator mode [12, 13] with coinciding atomic frequencies (AF), integrable models with explicit inter-atomic couplings have not been proposed. Likewise, though q-deformation, which physically signifies the introduction of anisotropy together with specific nonlinearity into the system, was considered for a few MR models [15, 16], their multi-atom and integrable

variants are not known. Therefore, it is indeed a challenge to find a scheme for generating integrable MR models with the desired properties.

To meet this challenge we construct a general integrable system based on the ancestor Lax operator of [14] and generate in a unified way a series of integrable multi-atom MR models with explicit inter-atomic interactions and nondegenerate AF. This includes such new generalizations for JC, BS, TI, models and discovers important integrable q-deformations such as qBS, qJC, qTI etc. It is worth noting that our integrable TI model exhibits full exponential nonlinearity without any approximation and multi-atom qBS and qJC models involve quantum group spin operators. Moreover, since our construction is based on a general Yang–Baxter (YB) algebra, together with the generation of various models at its different realizations, we can solve them exactly in a unified way through an algebraic Bethe ansatz (BA). Our strategy of construction is to start with a Lax operator by taking it as a combination $T(\lambda) = L^s(\lambda) \prod_j^{N_a} L_j^S(\lambda)$, with $L^s(\lambda)$ linked with the ancestor model of [14] and the N_a-number of $L_j^S(\lambda)$ related to the spin model [12]. By construction it must satisfy the YB equation $R(\lambda - \mu)T(\lambda) \otimes T(\mu) = (I \otimes T(\mu))(T(\lambda) \otimes I)R(\lambda - \mu)$, with a mutually commuting set of conserved operators obtained from the expansion $\tau(\lambda) = \mathrm{Tr}\, T(\lambda) = \sum_a C_a \lambda^a$ [17]. For standard MR models, as we will see below, the Lax operators are rational type linked with the simplest quantum R-matrix of the xxx spin chain [17], while for q-deformed models they are trigonometric type related to the R-matrix of the xxz chain [18]. We concentrate first on standard MR models and recall that in the rational case the 2×2 ancestor Lax operator may be given as

$$L^s(\lambda) = \begin{pmatrix} c_1^0(\lambda + s^3) + c_1^1 & s^- \\ s^+ & c_2^0(\lambda - s^3) - c_2^1 \end{pmatrix} \tag{1}$$

with operators **s** satisfying a quadratic algebra

$$[s^+, s^-] = 2m^+ s^3 + m^- \qquad [s^3, s^\pm] = \pm s^\pm \qquad [m^\pm, \cdot] = 0. \tag{2}$$

The central elements $m^\pm$ are expressed through arbitrary parameters appearing in (1) as $m^+ = c_1^0 c_2^0, m^- = c_1^1 c_2^0 + c_1^0 c_2^1$ and as is easy to see, their different choice reduces (2) to different algebras:

(i) $su(u)$, at $m^+ = 1, m^- = 0$ (ii) $su(1, 1)$, at $m^+ = -1, m^- = 0$

(iii) bosonic, at $m^+ = 0, m^- = -1$ (iv) canonical, at $m^+ = m^- = 0$ (3)

and the corresponding limits yield from (1) the respective Lax operators. In case (i), (1) reduces simply to the spin Lax operator

$$L_j^S(\lambda) = \begin{pmatrix} \lambda + S_j^z + c_j & S_j^- \\ S_j^+ & \lambda - S_j^z + c_j \end{pmatrix}. \tag{4}$$

Our Lax operator constructed as above would generate the set of all commuting conserved operators, with higher ones containing increasingly higher many-body interactions. The simplest among them is $C_{N_a} = s^3 + \sum_j^{N_a} S_j^z$, while the next in the set αC_{N_a-1} may be defined

as the Hamiltonian of our unified MR system:

$$H_{MR} = H_d + H_{Ss} + H_{SS}$$

$$H_d = \omega_f s^3 + \sum_j^{N_a} \omega_{aj} S_j^z$$

$$H_{Ss} = \alpha \sum_j^{N_a} \left(s^+ S_j^- + s^- S_j^+ + (c_1^0 + c_2^0) s^3 S_j^z \right)$$

$$H_{SS} = \alpha \sum_{i<j} \left((c_1^0 + c_2^0) S_i^z S_j^z + c_1^0 S_i^- S_j^+ + c_2^0 S_i^+ S_j^- \right).$$

(5)

Here H_{Ss} describes matter–radiation, while H_{SS}, matter–matter interactions. $\mathbf{S}_j$, $j = 1, 2, \ldots, N_a$ stand for an array of N_a atoms, each with $2s + 1$ levels and satisfy the $su(2)$ algebra. $\mathbf{s}$ on the other hand signifies a radiation or a vibration mode and satisfies more general algebra (2). In (5) the radiation frequency ω_f and the atomic frequencies ω_{aj}, $j = 1, 2, \ldots, N_a$ are defined through inhomogeneous parameters of the Lax operator as

$$\omega_f = \sum_j w_j \qquad w_j = \alpha(c_1^0 - c_2^0) c_j \qquad \omega_{aj} = \omega_f - w_j + \alpha(c_1^1 + c_2^1).$$

(6)

Remarkably, the general model (5) reduces to a new series of integrable multi-atom BS, JC and TI models in a unified way at the limits (ii), (iii) and (iv) of (3). For example, case (ii) with the choice

$$c_1^0 = -c_2^0 = 1, c_1^1 = c_2^1 \equiv c$$

(7)

yields from (5) the model

$$H_{BS} = \omega_f s^3 + \sum_j^{N_a} \left(\omega_{aj} S_j^z + \alpha(s^+ S_j^- + s^- S_j^+) \right) + \alpha \sum_{i<j}^{N_a} \left(S_i^- S_j^+ - S_i^+ S_j^- \right)$$

(8)

which with a bosonic realization of $su(1,1)$: $s^+ = \sqrt{N} b^\dagger$, $s^- = b\sqrt{N}$, $s^3 = N + \frac{1}{2}$ and the spin-s operator $\vec{S} = \frac{1}{2} \sum_k^{2s} \vec{\sigma}_k$, would represent a new *integrable multi-atom* BS *model* with inter-atomic interactions and different atomic frequencies. Note that at $N_a = 1$, when matter–matter interactions vanish and all AF coincide, (8) recovers the known model [13]. However, we solve below exactly through BA the more general case with nonvanishing interatomic couplings and all different AF: ω_{aj}, as defined in (6).

Similarly, a new *integrable multi-atom JC model* with matter–matter coupling is obtained from the same (5) under reduction (iii), consistent with $c_1^0 = \alpha$, $c_2^0 = 0$, $c_1^1 \equiv c$, $c_2^1 = -\alpha^{-1}$ and bosonic realization $s^- = b$, $s^+ = b^\dagger$, $s^3 = b^\dagger b$. We do not present here the explicit form of this easily derivable Hamiltonian, which yields the known model [12] only at $N_a = 1$, when interatomic couplings vanish and all AF become degenerate.

We can generate an *integrable TI model* with interatomic interactions, again from the same MR model (5) at reduction (iv), by fixing the parameter values as $c_1^0 = -1$, $c_1^1 \equiv c$, $c_2^0 = c_2^1 = 0$ and considering consistent realization through canonical variables as $s^\pm = e^{\mp ix}$, $s^3 = p + x$. We present here only its $N_a = 1$ form by a suitable combination with the other conserved quantity C_1:

$$H_{TI} = (\omega_a - \omega_f) S^z + S^{z2} + \alpha(e^{-ix} S^+ + e^{ix} S^-) + H_{xp}$$

(9)

with $H_{xp} = \frac{1}{2}(p^2 + x^2) + xp$, $\vec{S} = \frac{1}{2} \sum_k \vec{\sigma}_k$, which is a new integrable multi-atom TI model with full exponential nonlinearity without approximation.

For constructing *integrable q-deformed MR models* the strategy would be the same; only one has to start now from the trigonometric type ancestor Lax operator involving q-deformed operators and associated with the xxz R-matrix, the explicit form of which is given in [14]. For simplicity, we present here only the $N_a = 1$ case with the Hamiltonian

$$H_{q\text{MR}} = H_d + \left(s_q^+ S_q^- + s_q^- S_q^+\right)\sin\alpha$$
$$H_d = -\text{i}c_0\cos(\alpha X) + c\sin(\alpha X) \qquad X = \left(s_q^3 - S_q^z + \omega\right) \tag{10}$$

which represents a new class of MR models with $\mathbf{S}_q$ belonging to the quantum group $U_q(su(2))$ and $\mathbf{s}_q$ to a more general quantum algebra [14]. It is important to note that $\mathbf{s}_q$ can yield a variety of q-deformed operators, inducing (10) to generate a number of physically relevant q-deformed integrable MR models.

For example, an *integrable q-deformed BS model* may be constructed from (10) at $c_0 = 0$, by realizing $\mathbf{s}_q$ through the q-oscillator: $s_q^+ = \sqrt{[N]_q}b_q^\dagger$, $s_q^- = b_q\sqrt{[N]_q}$, $s_q^3 = N + \frac{1}{2}$, and quantum spin operator $\mathbf{S}_q$ by using its co-product [18] : $S_q^\pm = \sum_j^s q^{-\sum_{k<j}\sigma_k^z}\sigma_j^\pm q^{\sum_{l>j}\sigma_l^z}$, $S^z = \sum_j^s \sigma_j^z$. Note that at $s = 1$, we get an integrable version of an earlier model [15].

Similarly the same general model (10) with choice $c_0 = \text{i}, c = 1$ and realization $s_q^+ = b_q^\dagger, s_q^- = b_q, s^3 = N$ yield a new *integrable q-deformation of the JC model*, while under reduction $c_0 = \text{i}, c = 0$ and the same realization through canonical operators as for the TI model, it generates an *integrable q-deformation of the TI model*. By taking higher N_a values multi-atom integrable variants of all the above q-deformed matter–radiation models can be constructed.

We emphasize that all MR models presented here, similar to their unified construction, allow their exact BA solutions also in a unified and almost model-independent way. In the BA formalism the diagonal entries $\tau(\lambda) = T_{11}(\lambda) + T_{22}(\lambda)$ produce all conserved operators, while the off-diagonal elements $T_{21}(\lambda) \equiv B(\lambda)$ and $T_{12}(\lambda) \equiv C(\lambda)$ act like creation and annihilation operators of pseudoparticles with the M-particle state defined as $|M\rangle_B = B(\lambda_1)\cdots B(\lambda_M)|0\rangle$ and the pseudovacuum $|0\rangle$ through $C(\lambda)|0\rangle = 0$. The basic idea of algebraic BA [17] is to find the eigenvalue solution: $\tau(\lambda)|M\rangle_B = \Lambda(\lambda, \{\lambda_a\})|M\rangle_B$, for which diagonal elements $T_{ii}(\lambda), i = 1, 2$ are pushed through the string of $B(\lambda_a)$ towards $|0\rangle$, using the commutation relations obtainable from the YB equation. Considering further the actions $T_{11}(\lambda)|0\rangle = \alpha(\lambda)|0\rangle, T_{22}(\lambda)|0\rangle = \beta(\lambda)|0\rangle$, one arrives finally at the eigenvalue expression

$$\Lambda(\lambda, \{\lambda_a\}) = \alpha(\lambda)\prod_{a=1}^M f(\lambda - \lambda_a) + \beta(\lambda)\prod_{a=1}^M f(\lambda_a - \lambda) \tag{11}$$

where $f(\lambda)$ is defined through the elements of the R-matrix as $\frac{\lambda+\alpha}{\lambda}$, for the rational and as $\frac{\sin(\lambda+\alpha)}{\sin\lambda}$ for the trigonometric case. Expanding $\Lambda(\lambda, \{\lambda_a\})$ in powers of λ we obtain the eigenvalues for all conserved operators including the Hamiltonian, where the rapidity parameters $\{\lambda_a\}$ involved can be determined from the Bethe equations

$$\frac{\alpha(\lambda_a)}{\beta(\lambda_a)} = \prod_{b\neq a}\frac{f(\lambda_b - \lambda_a)}{f(\lambda_a - \lambda_b)} \qquad a = 1, 2, \ldots, M \tag{12}$$

which follow in turn from the requirement of $|M\rangle_B$ to be an eigenvector. Returning to our models we find that the major parts in key algebraic BA relations (11) and (12), described by R-matrix elements $f(\lambda)$, depend actually on the class to which the models belong, rather than on an individual model. Therefore, for all standard MR systems including BS, JC and TI models, $f(\lambda)$ is given by its same rational form, while for all q-deformed models such as qBS, qJC, qTI etc, by its trigonometric expression. The only model-dependent parts in these
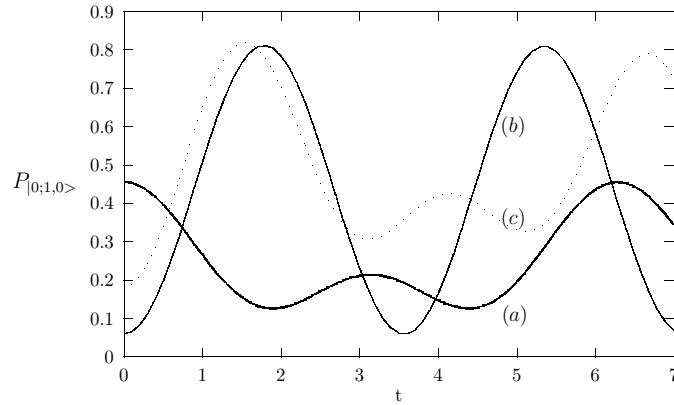
Figure 1. Time dependence of transition probability, showing Rabi oscillation at different detuning points: at (a) resonance, (b) degenerate detuning (c) further detuning with complex conjugate roots.

equations, expressed through $\alpha(\lambda)$ and $\beta(\lambda)$, are determined from our general Lax operator construction, which for the rational class using (1) and (4) is obtained as

$$
\begin{aligned}
\alpha(\lambda) &= \left(c_1^0(\lambda + r) + c_1^1 \right) \prod_{j}^{N_a} (\lambda - s + c_j) \\
\\
\beta(\lambda) &= \left(c_1^0(\lambda - r) - c_2^1 \right) \prod_{j}^{N_a} (\lambda + s + c_j)
\end{aligned}
\tag{13}
$$

where $r = \langle 0|s^3|0\rangle$, depends on the particular realization of (2) and $s = \langle 0|s^z|0\rangle$ denotes the atomic spin. Equation (13) yields easily the needed forms for BS, JC and TI models, at the corresponding choices of the parameters such as (7), as we have noted above. Similarly, for q-deformed models the quantum extension of (13) together with the trigonometric form for $f(\lambda)$ have to be considered. For the solution of TI and qIT models however one has to adopt a slightly different approach close to that of the Toda chain [19], since pseudovacuum is difficult to determine for such models.

For deriving physical consequences from our constructions, we consider the integrable two-level multi-atom BS model with inter-atomic couplings, by taking spin-$\frac{1}{2}$ operator $\vec{S}_j = \frac{1}{2}\vec{\sigma}_j$ in (8). Using the full strength of the BA method exact solutions for this multi-atom model with arbitrary N_a and different ω_a can be derived from the same BA relations (11)–(13) for all excitations, by just tuning the parameters involved to their required reduction (7). We demonstrate some novel features in Rabi oscillation and VRS by using the BA solutions in the $N_a = 2$ atom case of (8). The first excited energy spectrum $E_1 = \omega_f + 2\lambda_1$ linked with the cubic Bethe equation (12) gives three distinct real roots along the resonance line $\omega_f = \omega_a(\equiv \omega_{a1} = \omega_{a2})$, resulting in a triplet structure in the VRS with splittings in excitation spectrum: $E_1 = 2.02, 3.02, 4.02$ for $\omega_f = 3.02, \alpha = 1$. Consequently, the Rabi oscillation becomes involved (see figure 1(a)) with three transition frequencies. For small detuning $\delta = \omega_f - \omega_a$ the roots remain real, while at $\delta = \mp 0.30$ two of them coincide, collapsing the Rabi-splitting to the usual doublet. The excitation spectrum correspondingly reduces to $E_1 = 2.05, 3.81$ (degenerate) at sub-detuning and to $E_1 = 2.22$ (degenerate), 3.99 at super-detuning points, reducing the Rabi-oscillations to single frequency mode (figure 1(b)). Beyond

these detuning points two of the roots become complex conjugates, leading to an irregular Rabi oscillation (figure 1(c)).

Higher excitations for this model as well as its N_a-atom extension (8) can be solved exactly following the standard BA formalism presented above. We explore now some subtle points and apparent controversies regarding the BA solution of BS models, which also have relevance for other integrable models. A common belief, though proved only for specific models [20], is that the degeneracy condition (i.e. $\lambda_a = \lambda_b$) for the Bethe states: $\frac{\alpha(\lambda_1)}{\beta(\lambda_1)} = \pm 1$ cannot be solved apparently for any integrable model. We however find that for our multi-atom BS model at the resonance point $\delta = \omega_f - \omega_a = 0$, the degeneracy condition, which is equivalent to $\alpha(\lambda_1) = \beta(\lambda_1) = 0$ is indeed fulfilled, yielding a nontrivial solution $\lambda_1 = \frac{1}{2}(-\omega_f \pm 1)$, which recovers as well the known spectrum for the standard BS model: $E_M = 2M(\omega_f + \lambda_1) = ME_1$, at resonance [6].

Another apparent controversy regarding multi-atom BS models, which is also generic for many other models but not emphasized properly in the literature, arises due to the fact that the dimension of the underlying Hilbert space for these models with N_a number of two-level atoms has an upper bound 2^{N_a}, for a fixed pseudoparticle number. For the standard BS model it is just 2. Therefore, the complexity of the problem cannot increase further for higher excitations with $M > N_a$. For example, in the BS model by diagonalizing the Hamiltonian directly one can easily get the exact energy spectrum for arbitrary excitation [6]. However, when we try to solve the same problem through the Bethe ansatz the solution must become increasingly complicated for higher excitations $|M\rangle_B$, since one has to find all M Bethe roots $\{\lambda_a\}, a = 1, 2, \ldots, M$ as solutions to general Bethe equations (12), which is impossible analytically! We resolve this problem in an intriguing way by observing that Bethe state $|M\rangle_B$ and the energy eigenvalues depend in fact not on M number of roots λ_a individually, but only on some symmetric combinations of them and moreover, the number of these relevant variables does not exceed the dimension of the Hilbert space. For the BS model for example, we find them to be only two: X_M, Y_M and for deriving them explicitly we introduce an equivalent set of symmetric Bethe roots through symmetric combinations of the original M roots[1]: $s_1 = \sum_a \lambda_a, s_2 = \sum_{ab} = \lambda_a \lambda_b, \ldots, s_M = \prod_a \lambda_a$. Combining suitably BAE (12) and (11) for the BS model, expressed through symmetric roots we arrive at the equations

$$E_M X_M = \Delta_+ X_M + M Y_M \qquad E_M Y_M = \Delta_- Y_M + M X_M \qquad (14)$$

where $\Delta_\pm = M\omega_f \pm \frac{1}{2}\delta$. On the other hand, expressing the Bethe states through two basic states of the model we find $|M\rangle_B = X_M|M, -\rangle + Y_M|M - 1, +\rangle$, i.e. dependent again on the above two relevant variables only. It is easy to see that the action of the BS Hamiltonian on this eigenstate also reproduces the same relation (14), we have derived from the BAE. Fixing $\omega_f = 1$ for simplicity, we find these two variables explicitly through symmetric Bethe roots in the form $X_M = M s_M$ and $Y_M = -M s_M + \frac{1}{2}\delta(s_{M-1} + \cdots + s_1 + 1)$. From the first of the relations (14) we get the energy spectrum as $E_M = \Delta_+ + M\kappa_M$, where $\kappa_M = \frac{Y_M}{X_M}$ and using both these relations we derive the simple equation $M\kappa_M^2 + \delta\kappa_M - M = 0$. This quadratic equation is solved easily to yield $E_M = M\omega_f \pm (\delta^2 + M^2)^{\frac{1}{2}}$, recovering the known spectrum of the BS model in the general $\delta \neq 0$ case. Thus through BA we get the explicit result for all higher excitations, also analytically, resolving the raised controversy. Similar arguments must hold in the corresponding problem for other models.

Thus we have proposed through general Yang–Baxter algebra a series of new integrable multi-atom matter–radiation models including q-deformed models and solved them exactly

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[1] A useful reformulation of Bethe equations achieved through *symmetric roots* will be presented elsewhere.

through the Bethe ansatz in a unified way. The integrable trapped ion (TI) and q-deformed TI models introduced here are new, while q-deformed Jaynes–Cummings (JC) and Buck–Sukumar (BS) models are multi-atom as well as integrable extensions of earlier models [15, 16]. The proposed JC and BS models are nontrivial generalizations of well-known models [12, 13], with the inclusion of inter-atomic interactions and nondegenerate atomic frequencies. We find that in contrast to the popular belief, the degenerate Bethe states do exist in the multi-atom BS models at the resonance point. Multi-radiation modes can be included easily in such models preserving integrability. Identifying the models in real systems and experimental verification of the related results presented here, especially in many-atom microlasers [21], would be an important problem.

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